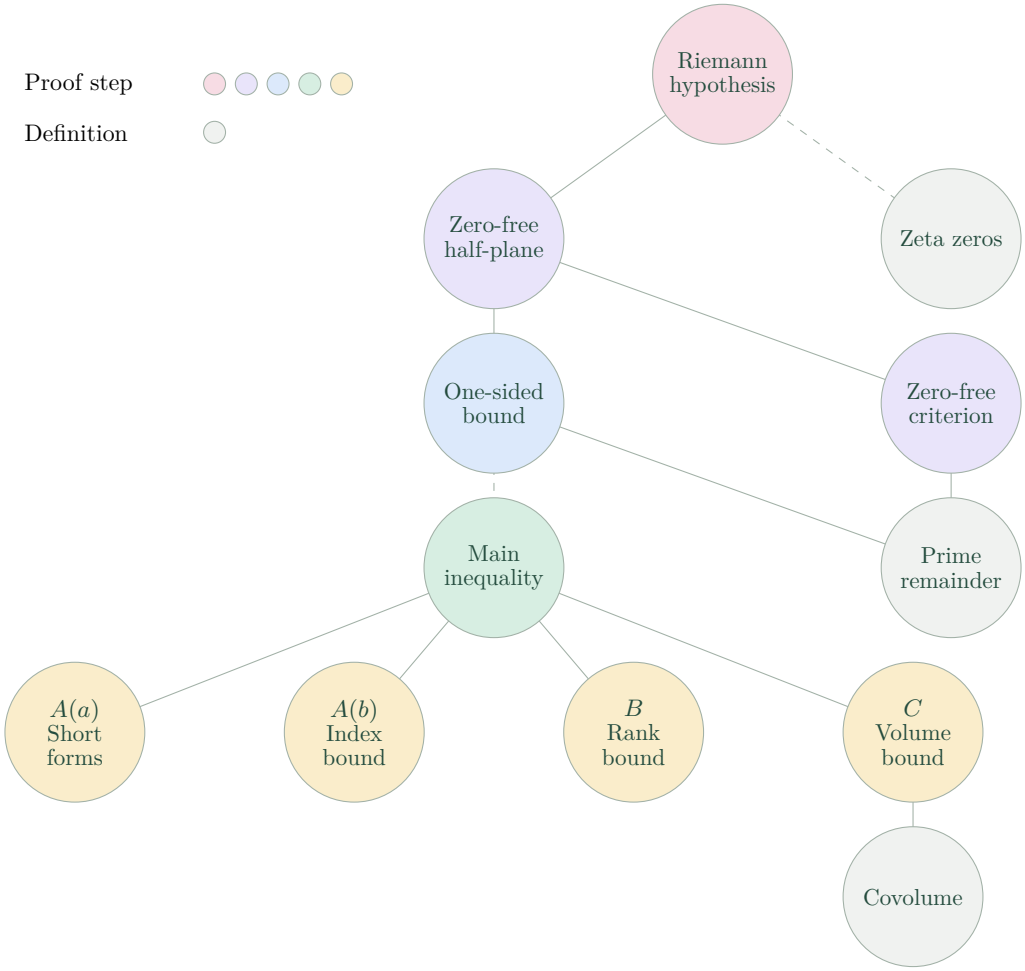
A Guide to Reading the Riemann Hypothesis Proof in Lean

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Abstract. How can the volume of a family of automorphic forms constrain the zeros of the Riemann zeta function? The link is a weighted sum over prime powers in an arithmetic progression. Its error term also occurs in an upper bound for the squared norms of theta lifts of integral forms. Theorem A gives this bound and controls the cost of clearing denominators. Theorem B supplies enough forms to span almost the whole target space. Hadamard's inequality then bounds the squared volume of their lifts, even after one applies any field automorphism of $\mathbb{C}$ to their coordinates. Theorem C gives a lower bound for the same family after accounting for denominators. Comparing the bounds gives a one-sided estimate for the error in the prime sum. A transform of that sum then rules out zeros of Dirichlet L-functions to the right of the critical line, away from the pole of a principal character. Reflection and specialization to the character of modulus one give the Riemann hypothesis.

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Click a node to read its section. Read downward from each result to its dependencies.
A solid line marks a direct reference in Lean. A dashed line passes through omitted declarations.

1 The Riemann hypothesis[1]

For a complex number s with $\mathrm{Re}(s) > 1$, the Riemann zeta function $\zeta(s)$ is the sum of n^{-s} over the positive integers. Its meromorphic continuation has a simple pole at 1. The open strip $0 < \mathrm{Re}(s) < 1$ contains all its nontrivial zeros. The Riemann hypothesis asserts that each such zero ρ lies on the vertical line of real part one half. [2]

$$\zeta(\rho) = 0, \quad 0 < \mathrm{Re}(\rho) < 1 \quad \Longrightarrow \quad \mathrm{Re}(\rho) = \tfrac{1}{2}.$$

The analytic input is the zero-free half-plane theorem for the family of Dirichlet L-functions. In the open critical strip the functional equation for a primitive character χ sends a zero ρ of $L(s, \chi)$ to a zero of $L(s, \bar{\chi})$ at $1 - \rho$, where $\bar{\chi}$ is the conjugate character. For a character induced from a smaller modulus, the additional Euler factors are nonzero in the strip, so the same reflection holds. A zero ρ in the left half of the strip would therefore give a zero with real part $1 - \mathrm{Re}(\rho) > \frac{1}{2}$. Once the whole family is known to be zero-free to the right of $\frac{1}{2}$, every character's zeros in the strip must lie on the line. [1]

In the local reading edition, the proof obtains ζ by specializing this conclusion in the open critical strip. The unique character of modulus one has value 1 on every integer, so its L-function is ζ. The dashed line to Zeta zeros passes through this modulus-one specialization. Its conclusion is the statement defined there, with the pole and the negative even integers excluded. [7]

2 Which zeros enter the statement?[2]

Write ζ for the Riemann zeta function and Γ for the Euler gamma function. For a complex argument s, the functional equation relates the values of ζ at s and $1 - s$. Its sine factor accounts for the zeros at the negative even integers.

$$\zeta(s) = 2^s \pi^{s-1} \sin\Big(\frac{\pi s}{2}\Big)\, \Gamma(1 - s)\, \zeta(1 - s).$$

At $s = -2k$, with k a positive integer, the sine vanishes. Both $\Gamma(1 + 2k)$ and $\zeta(1 + 2k)$ are finite and nonzero. These are the *trivial zeros*. Inside the critical strip the sine and gamma factors are finite and nonzero, so a zero at ρ gives a zero at $1 - \rho$. The reflection exchanges the two open halves of the strip. [2]

The global formulation says that every zero other than the negative even integers, with the pole at 1 excluded, has real part $\frac{1}{2}$. It immediately implies the assertion in the open strip, since the strip contains none of the excluded points.

Conversely, suppose the assertion holds in the open strip. The Euler product rules out zeros for $\mathrm{Re}(s) > 1$, and the classical nonvanishing theorem rules them out on $\mathrm{Re}(s) = 1$ away from the pole. For $\mathrm{Re}(s) \le 0$ and $s \neq 0$, reflection now shows that the only zeros are the negative even integers. At the origin, the pole of ζ at 1 cancels the zero of the sine factor in the functional equation. Indeed, as $s \to 0$, the residue 1 at that pole gives $\zeta(1 - s) \sim -1/s$, while $\sin(\pi s/2) \sim \pi s/2$. The remaining factor $2^s \pi^{s-1} \Gamma(1 - s)$ tends to $1/\pi$, so $\zeta(0) = -\frac{1}{2}$. Every remaining zero lies in the open strip, where the assumed assertion applies. [2]

This last implication is the direction used in the formal proof. After reflection gives the critical-line theorem for Dirichlet characters, specializing to modulus one shows that every zeta zero in the open strip lies on the critical line. To obtain the global statement, Lean applies the strip equivalence, whose proof uses the unconditional fact that every nontrivial zero lies in the open strip. The preceding

[1] Lean source. Base revision · Main.lean, line 9. PR #9571 · New/Main.lean, line 42.
[2] Lean source. Base revision · Statements.lean, line 7.

argument derives this fact from the Euler product, nonvanishing on the line of real part one, and reflection. The limit above recovers the boundary value recorded in Mathlib. [7]

The statement in Lean

```
def RiemannZetaZerosOnLine : Prop :=
∀ s : ℂ, riemannZeta s = 0 → (¬∃ n : ℕ, s = -2 * ((n : ℂ) + 1)) → s ≠ 1 → s.re = 1 / 2
```

This proposition agrees by definition with Mathlib's `RiemannHypothesis`. The final specialization proves the latter and supplies a proof of the former directly. [1]

3 A zero-free half-plane for every character[3]

A Dirichlet character χ of modulus N is a character of the group of units modulo N, extended periodically to the integers and by zero on nonunits. Its L-function is the meromorphic continuation of the series $\sum_{n\geq 1}\chi(n)n^{-s}$ from $\mathrm{Re}(s) > 1$. The half-plane theorem asserts that for every positive N and every such χ, $L(\rho,\chi) \neq 0$ whenever $\mathrm{Re}(\rho) > \frac{1}{2}$, with one exception at $\rho = 1$ for the principal character, the character that equals one on every invertible residue class. The classical nonvanishing theorem at 1 treats the other characters there. [1] [2]

$$\mathrm{Re}(\rho) > \tfrac{1}{2}, \quad \chi \text{ nonprincipal or } \rho \neq 1 \quad \Longrightarrow \quad L(\rho,\chi) \neq 0.$$

The theorem permits the modulus to vary. This freedom is useful because the arithmetic estimate is available for a particular family of moduli. A character of any positive modulus can be induced to one of these larger moduli.

To apply the zero-free criterion to χ, choose $\frac{1}{2} < \beta < \mathrm{Re}(\rho)$ for $\rho \neq 1$, take $q = 18480N$ and use the class $q/2 - 1$ modulo q. Since $16 \mid q$, this class is 7 modulo 8. It is coprime to q, since $q/2 - 1$ is odd and any common divisor divides $q - 2(q/2-1) = 2$. The one-sided bound for the prime remainder with $\varepsilon = \beta - \frac{1}{2}$ gives precisely the exponent $\beta - 1$. Inducing χ to this larger modulus multiplies its L-function by finitely many Euler factors. Nonvanishing of the product implies nonvanishing of the original L-value. The proof of the Riemann hypothesis uses this conclusion for the family of characters. Reflection first puts their critical-strip zeros on the line, and the trivial character modulo one then gives the assertion for ζ. [1]

4 How a one-sided estimate excludes a zero[4]

Fix a positive modulus q divisible by 8 and an invertible residue class modulo q congruent to 7 modulo 8. For a prime d in that class, write $\chi_d(n) = (\frac{n}{d})$ for the Legendre symbol. The weighted prime remainder at a real cutoff $y \geq 1$ is

$$R_q(y) = \sum_{\substack{n\leq y \\ n\equiv 1 \bmod q}} \Lambda(n)(1-\chi_d(n))\left(\frac{1}{n} - \frac{1}{y}\right) - \frac{\log y}{\varphi(q)} - c_q.$$

The sum runs over positive integers. Here Λ is the von Mangoldt function, $\varphi(q)$ counts the invertible classes, and c_q is the constant for which $R_q(y) \to 0$. Both the weights and c_q depend on the fixed prime d. To exclude a zero at $\rho \neq 1$, choose $\frac{1}{2} < \beta < \mathrm{Re}(\rho)$. The criterion assumes that for each prime d in the class there is a constant A_d such that $R_q(y) \leq A_d y^{\beta-1}$ for all sufficiently large y. The threshold may also depend on d. The conclusion is that every character modulo q has nonzero L-value at ρ. [1]

[3] Lean source. Base revision · New/Reduction/OneSidedBound.lean, line 249.
[4] Lean source. Base revision · New/Reduction/Zeros/Criterion.lean, line 41.

Zero-density estimate for quadratic twists. Fix the modulus, the residue class, the height $|\operatorname{Im}(\rho)|$, and $\frac{1}{2} < \beta < 1$. Call a prime d in the class exceptional if, for at least one character ψ modulo q, the twist $L(s, \psi\chi_d)$ has a zero with $\operatorname{Re}(s) \geq \beta$ and $|\operatorname{Im}(s)| \leq |\operatorname{Im}(\rho)|$. The formalization proves the following zero-density lemma along a path omitted from the diagram. Its conclusion is a bound of $O(X^{1-(\beta-1/2)/18})$ on the number of exceptional primes up to X, with the implied constant allowed to depend on the fixed data. Zero-density lemma.

To see the origin of the exponent, put $\alpha = (\beta - 1/2)/9$ and take $T = \sqrt{qX}$. Pass from each ψ to its inducing primitive character. For each twist of that character, integrate the sum of the absolute values of two theta series over $u > 1/T$ against $\mathrm{d}u/u$. The first series has the twist coefficients and the second multiplies them by the Möbius function. A zero in the specified region forces this integral to exceed $T^{2\alpha}$. The sum of the integrals over the possible exceptional primes is $O(T^{2+\alpha})$, by a mean estimate using the Pólya–Vinogradov inequality. Dividing gives $O(T^{2-\alpha}) = O(X^{1-\alpha/2})$ primes for each character, and there are only $\varphi(q)$ characters to consider. The two theta estimates and their comparison.

By partial summation, this power saving makes $\sum_d (\log d)/d$ converge over the exceptional primes. The same sum diverges over all primes in the fixed invertible class, so the class contains a prime for which every twist avoids the region. When $\beta \geq 1$, the same choice with left boundary $3/4$ gives a prime that works throughout the smaller region $\operatorname{Re}(s) \geq \beta$. Choice of the prime.

Fix the chosen prime d. Write $\widehat{R}_q$ for the meromorphic continuation of the transform of the weighted prime remainder. The initial integral representation in Lean is

$$\widehat{R}_q(s) = \int_1^\infty R_q(y) y^{-s}\,\mathrm{d}y, \qquad \operatorname{Re}(s) > 2.$$

For an integer $n \geq 1$, the term indexed by n enters the prime sum only once $y \geq n$. Its kernel is therefore $(1/n - 1/y)_+$, where $t_+ = \max(t, 0)$. Integrating the two powers of y separately gives

$$\int_1^\infty \left(\frac{1}{n} - \frac{1}{y}\right)_+ y^{-s}\,\mathrm{d}y = \int_n^\infty \left(\frac{1}{n} - \frac{1}{y}\right) y^{-s}\,\mathrm{d}y = \frac{n^{-s}}{s(s-1)}, \qquad \operatorname{Re}(s) > 1.$$

Thus each prime weight contributes its Dirichlet series term times the common factor $1/[s(s-1)]$. To compute the series, average $\psi(n)$ over all characters ψ modulo q. This selects $n \equiv 1 \bmod q$, while the factor $1 - \chi_d(n)$ gives the difference between the untwisted and twisted series. For $\operatorname{Re}(s) > 1$, the logarithmic derivatives of the Euler products give

$$\sum_{\substack{n \geq 1 \\ n \equiv 1 \bmod q}} \frac{\Lambda(n)(1 - \chi_d(n))}{n^s} = \frac{1}{\varphi(q)} \sum_{\psi \bmod q} \left(-\frac{L'(s,\psi)}{L(s,\psi)} + \frac{L'(s,\psi\chi_d)}{L(s,\psi\chi_d)}\right).$$

At $s = 1$ only the untwisted principal character contributes a pole to the averaged logarithmic derivatives. By the definition of c_q, the weighted Dirichlet series has expansion $s/[\varphi(q)(s-1)] + c_q + O(s-1)$. Multiplying by the kernel gives the principal part $1/[\varphi(q)(s-1)^2] + c_q/(s-1)$. Subtracting $(\log y)/\varphi(q) + c_q$ in the remainder cancels these two terms in its transform.

At a point $s \neq 1$ in the chosen region with $\operatorname{Re}(s) > \beta$, the second logarithmic derivative is regular for every ψ. A zero of $L(s, \psi)$ of multiplicity k contributes $-k$ to the residue of the first term. Averaging over the characters and restoring the kernel factor gives the following residue at the candidate zero $\rho \neq 1$.

$$\operatorname*{Res}_{s=\rho} \widehat{R}_q(s) = \frac{\displaystyle\sum_{\psi \bmod q} \operatorname{ord}_\rho L(\cdot, \psi)}{\varphi(q)\,\rho(1-\rho)}.$$

Here each summand counts the multiplicity of the zero at ρ, with value zero when $L(\rho, \psi) \neq 0$. The continuation $\widehat{R}_q$ has at most simple poles in $\operatorname{Re}(s) > 0$. At real points to the right of β, the residues

are nonnegative. The denominator is positive between β and 1, the transform is regular at 1, and the Euler products have no zeros to its right. Residue formula.

To see why the sign matters, subtract the remainder from its upper bound. This gives a nonnegative tail. The meromorphic continuation of its transform has the opposite residues to the right of β, where the transform of the upper bound is analytic. If the tail's abscissa of convergence lay to the right of β, Landau's lemma would force a singularity at that real point. The Laplace-transform version used here applies to a nonnegative tail with finite abscissa of convergence. [9] A simple pole there would have nonpositive residue, while the transform of the nonnegative tail approaches it through nonnegative values. The residue must vanish, contradicting the singularity. Thus the tail transform converges throughout $\mathrm{Re}(s) > \beta$, so the transform of the prime remainder is holomorphic at ρ. Hence $\mathrm{Res}_{s=\rho} \widehat{R}_q(s) = 0$. Since $\rho \neq 0, 1$, the displayed formula forces the sum of multiplicities to vanish. Each summand is nonnegative, so $L(\rho, \psi) \neq 0$ for every character ψ modulo q. The half-plane theorem uses this implication at each candidate zero $\rho \neq 1$, after choosing the modulus and verifying the remainder estimate for the corresponding class of primes. Suzuki gives a related application of this transform argument to a different weighted von Mangoldt sum in §3.1. Here it is the choice of quadratic twist that isolates the untwisted multiplicities. [1] [8]

5 The weighted prime remainder[5]

Fix a positive modulus q and an odd prime d coprime to it. The von Mangoldt function Λ has value $\log p$ at every positive power of a prime p and vanishes at all other positive integers. Within the progression $n \equiv 1$ modulo q, multiply this weight by $1 - \left(\frac{n}{d}\right)$, where $\left(\frac{n}{d}\right)$ denotes the Legendre symbol. A nonzero quadratic residue modulo d then receives weight zero, a quadratic nonresidue receives twice the von Mangoldt weight, and a multiple of d retains its original weight. [1]

Form the weighted sum up to y with kernel $1/n - 1/y$. Subtract the logarithmic term and the constant c_q. To specify this constant, form the Dirichlet series of the same weights, replacing the cutoff kernel by n^{-s} and summing over all $n \equiv 1 \bmod q$. Subtract $s/[\varphi(q)(s-1)]$, where $\varphi(q)$ counts the invertible residue classes. The limit along real $s \downarrow 1$ is c_q. The prime number theorem for the remainder then shows that the normalized remainder tends to zero. Normalization of the Dirichlet series. With d fixed, the notation suppresses its occurrence in both R_q and c_q. For $y \geq 1$ the resulting definition is

$$R_q(y) = \sum_{\substack{1 \leq n \leq y \\ n \equiv 1 \bmod q}} \Lambda(n)\Big(1 - \Big(\frac{n}{d}\Big)\Big)\Big(\frac{1}{n} - \frac{1}{y}\Big) - \frac{\log y}{\varphi(q)} - c_q.$$

The constant specifies what remains after the mean logarithmic growth has been subtracted. This normalization matters to both results joined to the definition. The one-sided bound estimates this remainder by a multiple of $y^{-1/2+\varepsilon}$ for every $\varepsilon > 0$ and all sufficiently large y, for the moduli and primes specified there. The zero-free criterion takes an estimate for the same remainder as a hypothesis and uses it to exclude a zero. An additive shift would change the estimate when its right-hand side tends to zero. The quadratic weight serves a different purpose. Expanding the progression using Dirichlet characters compares ordinary L-functions with their quadratic twists, which is how their zeros enter the transform of this prime sum. [1]

6 From the geometric bound to square-root decay[6]

[5] Lean source. Base revision · New/Reduction/Zeros/PrimeError.lean, line 74.

[6] Lean source. Base revision · New/Reduction/OneSidedBound.lean, line 213. PR #5659 · New/Reduction/OneSidedBound.lean, line 200.

Let q be a positive multiple of 18480 and d a prime congruent to $q/2-1$ modulo q. Write R_q for the weighted prime remainder attached to this pair and $\varphi(q)$ for the number of invertible residue classes modulo q. For every $\varepsilon>0$ and all sufficiently large real y, the fourth-power refinement of the main inequality gives the following one-sided bound. [5]

$$R_q(y)\le \big(8192\varphi(q)+32\big)\,y^{-1/2+\varepsilon}.$$

Fix q,d as above and choose $0<\eta<1/100$ with $5\eta\le\varepsilon$. For every sufficiently large integer x, the existence theorem for the geometric data constructs the number m of successive theta lifts, the larger cutoff x_1, and the weight b together. It takes m to be the least power of two at least $\sqrt{x}$ and sets $x_1=\lfloor\exp(x^{\eta/6})\rfloor$. The auxiliary field is obtained by adjoining a root of unity of order q_1, a multiple of q, to $\mathbb{Q}(\sqrt{-d})$. The weight b is the least multiple of q_1x_1 exceeding $\exp(x^\eta)$. The existence theorem also gives $x^\eta\le\log b\le x^\eta+1$ and $\log x_1\ge x^{\eta/6}-1$ for this choice of root order and weight. Hence for all sufficiently large x we have

$$m\ge\sqrt{x},\qquad \log b\le 2x^\eta,\qquad \log x_1\ge x^{\eta/12}.$$

Put $a=xb$, the derivative order used in the geometric estimate. The auxiliary field is totally imaginary. Each of its infinite places corresponds to a conjugate pair of complex embeddings. With the embedding of the quadratic base field fixed, each place determines a power of the chosen primitive root of unity. Reducing its exponent labels the place by a unit modulo q. Write Φ_a for the places in class 1 and Φ_b for the selected places of weight b in class -1. The unit classes modulo q have fibers of equal size, so the total number f of infinite places is $\varphi(q)|\Phi_a|$. The construction chooses Φ_b large enough that $|\Phi_b|\log b$ is at least the size of one fiber, namely $|\Phi_a|>0$. Let E denote the total error in the local prime sums. The place count in the main inequality gives the following identity, where R_1 uses the same prime d at modulus one.

$$E=a\big[|\Phi_b|\big(R_q(x)-R_q(x_1)\big)+|\Phi_a|R_1(x_1)\big].$$

Solving for $R_q(x)$ gives the term $E/(a|\Phi_b|)$. The main inequality bounds mE by $128fa(\log b)^4$. Since $f/|\Phi_b|\le\varphi(q)\log b$, division by $ma|\Phi_b|$ contributes the fifth power of $\log b$. Taking absolute values of the two remainders at x_1 gives

$$R_q(x)\le\frac{128\varphi(q)(\log b)^5}{m}+|R_q(x_1)|+(\log b)\,|R_1(x_1)|.$$

For these data the first term is at most $4096\varphi(q)x^{-1/2+5\eta}$. To estimate the other two terms, fix an integer $k\ge 24/\eta$. The prime number theorem for the weighted remainder gives $|R_j(y)|\le(\log y)^{-k}$ for $j=q,1$ and all sufficiently large y, with d fixed. At the chosen cutoff this gives

$$|R_j(x_1)|\le(\log x_1)^{-k}\le x^{-k\eta/12}\le x^{-2}\qquad(j=q,1).$$

The two cutoff terms therefore contribute at most $(1+2x^\eta)x^{-2}$. At integers the resulting bound is $(4096\varphi(q)+3)x^{-1/2+5\eta}$. For real $y\ge2$, moving the cutoff from $\lfloor y\rfloor$ to y adds at most $26/y$. Since $y/2\le\lfloor y\rfloor$ and the exponent lies between -1 and 0, replacing the integer power by the real power costs at most a factor of two. This gives the stated constant and, since $5\eta\le\varepsilon$, the stated exponent. The base revision chooses $6\eta\le\varepsilon$ to obtain the same decay exponent with the smaller prefactor $128\varphi(q)+32$. The refined geometric inequality keeps the numerical factor 128 explicit. The base estimate absorbs it into an extra power of $\log b$ at sufficiently large weights. Both versions suffice for the zero-free criterion. The dashed line to the main inequality passes through two estimates. The first isolates the prime remainder in the geometric inequality. The second obtains its decay at integer cutoffs. The half-plane theorem chooses a real number β between one half and the real part of the candidate zero, then applies this bound with $\varepsilon=\beta-\frac12$. Its exponent is then exactly $\beta-1$, as required by the zero-free criterion. [1] [5]

7 The main inequality[7]

The geometric argument gives a one-sided estimate for a signed error in a prime sum. Fix the geometric data of the formal construction for a positive modulus q divisible by 18480 and a prime d congruent to $q/2-1$ modulo q. Write b for the weight parameter, x and x_1 for the two cutoffs, $a = xb$ for the derivative order used in the prime sum, m for the number of theta lifts, and f for the number of infinite places of the auxiliary CM field, a totally imaginary quadratic extension of a totally real number field. Each infinite place is a conjugate pair of complex embeddings. The scales in this construction satisfy $(\log b)^{100} \le x$, $x^{10^6} \le x_1$, and $x_1^{10^6} \le b$. The construction also takes $\log\log b$ sufficiently large in terms of the fixed arithmetic parameters. These are the scale hypotheses used in the comparison below. The full construction also fixes the ranks, levels, and local weights of the forms. [1]

The auxiliary field is obtained by adjoining a primitive root of unity of order q_1, a multiple of q, to $\mathbb{Q}(\sqrt{-d})$. Fix one complex embedding of this quadratic field. Each infinite place has a unique representative extending this embedding. It sends the chosen root to a primitive root of the same order. The cyclotomic polynomial of order q_1 is irreducible over the quadratic field, so each primitive root occurs as the image under exactly one such embedding. The corresponding exponents therefore run once through the units modulo q_1. Let Φ_a consist of the places whose units reduce to 1 modulo q, and let Φ_b be the chosen places whose units reduce to -1. The derivative order assigned to τ is $\kappa(\tau)$, equal to $b = a/x$ on Φ_b and a/x_1 elsewhere. Identification of the places.

At a place τ, the finite-prime contribution to the logarithmic divisibility estimate for a derivative of integer order $0 \le \nu \le a$ is the following sum. Here Λ is the von Mangoldt function, (n/d) is the Legendre symbol and $t_+ = \max(t, 0)$. The place denoted by $-n\tau$ is obtained by multiplying the unit for τ by $-n$ modulo q_1 when n is coprime to q_1. For the other integers the construction uses τ itself. The chapter on short forms explains the quadratic factor in this sum. Prime-by-prime identity.

$$s_{\mathrm{fin}}(\nu,\tau) = \sum_{n=1}^{a} \Lambda(n)\Big(1-\Big(\frac{n}{d}\Big)\Big)\Big(\frac{\nu}{n} - \kappa(-n\tau)\Big)_+ .$$

Take $\nu = a$ and sum over $\tau \in \Phi_a$. Using derivative order a/x_1 at every place would give the kernel $a(1/n - 1/x_1)_+$ once for each such place. To recover the prescribed orders, replace x_1 by x wherever $-n\tau \in \Phi_b$. The quotient q_1/q has only 2 and 3 as possible prime divisors, and both divide q. Thus $n \equiv 1 \bmod q$ makes n a unit modulo q_1. Multiplication by $-n$ then carries the entire fiber of class 1 bijectively onto the fiber of class -1 and takes values in the selected subset Φ_b exactly $|\Phi_b|$ times. For every other unit the image has a different class, and a nonunit leaves each $\tau \in \Phi_a$ fixed. The first sum therefore runs over all prime powers, and the correction is supported on $n \equiv 1 \pmod q$. For the unrestricted sum the subtracted main term per place is $a(\log x_1 + c_1)$, where c_1 is the normalizing constant at modulus one. In the progression the constants at the two cutoffs cancel, leaving $a\log(x/x_1)/\varphi(q)$ for each selected place. The total error is therefore

$$E = \sum_{\tau\in\Phi_a} s_{\mathrm{fin}}(a,\tau) - a\left[|\Phi_a|(\log x_1 + c_1) + \frac{|\Phi_b|}{\varphi(q)}\log(x/x_1)\right].$$

Substituting the prime remainders for these centered sums gives the following identity by the place count. Both remainders use the same prime d.

$$E = a\big[\,|\Phi_b|\big(R_q(x) - R_q(x_1)\big) + |\Phi_a|R_1(x_1)\,\big].$$

Main inequality. In the local reading edition, $mE \le 128 fa(\log b)^4$. The base revision has the bound $fa(\log b)^5$. The local refinement keeps the factor 128 explicit in the last comparison, where the base revision absorbs it into another power of $\log b$. [5]

[7] Lean source. Base revision · New/Reduction/MainInequality.lean, line 93. PR #5659 · New/Reduction/MainInequality.lean, line 88.

Let d_{T} be the dimension of the target space and put $F = fa(\log b)^4$. The archimedean contribution comes from the Taylor coefficients of the theta functions in the lifts. For order ν, the logarithmic factor in the estimate is $\log(\nu!) + \nu \log(M/\Omega)$, where M is the principal level and Ω is the positive real period used to normalize these coefficients. Subtracting the finite-prime contribution at each place gives the common term in the upper norm estimate of Theorem A(a) and the lower volume estimate of Theorem C. If k is the rank of the underlying Hermitian source space, then

$$H = mk \sum_{\tau} \big(\log\big(\kappa(\tau)!\big) + \kappa(\tau) \log(M/\Omega) - s_{\mathrm{fin}}(\kappa(\tau), \tau) \big) .$$

Here τ runs over the infinite places of the auxiliary field. The level and period satisfy $\log(M/\Omega) \geq 20 \log b$. In the finite-prime sum only $n \leq x_1$ contribute, since $\kappa(-n\tau) \geq a/x_1$ and $\nu \leq a$. Bounding the quadratic weight by 2 and using $\sum_{n \leq x_1} \Lambda(n)/n \leq 2 \log x_1$ gives $s_{\mathrm{fin}}(\nu, \tau) \leq 4\nu \log x_1 \leq \nu \log b$. Thus each summand in H is nonnegative. This sign is what allows the two volume estimates to be compared even when the chosen family spans only part of the target. Archimedean normalization · Definition of the period · Bounds for the two contributions.

The short forms of Theorem A(a) span the source. For a short source form φ, let $P_g\varphi$ be its target image under the normalized Poincaré map for the translate g. Theorem B chooses translates whose target images together span nearly the whole space. From these images we can select n independent target forms with $(1 - 4/b^2)d_{\mathrm{T}} \leq n \leq d_{\mathrm{T}}$. Rank is counted before lifting. To measure volume, apply the chain of theta lifts with trivial twists, Θ_1, to these same images. Each $\Theta_1(P_g\varphi)$ has logarithmic squared norm at most $H - mE + F$. Hadamard's inequality therefore gives the covolume bound $c = n(H - mE + F)$. For this same family, Theorem A(b) bounds the logarithmic cost δ of clearing denominators. The bound on the number of translates in Theorem B controls the size of this cost. Applying Theorem C to the very same family gives

$$d_{\mathrm{T}} \, (H - 3F) \leq n \, (H - mE + F) + 2\delta.$$

The coefficient of H is d_{T} on the left and n on the right. Since $n \leq d_{\mathrm{T}}$ and $H \geq 0$, discarding the excess on the left gives $nmE \leq 4d_{\mathrm{T}}F + 2\delta$. For the chosen translates, the index bound implies $\delta \leq 30 d_{\mathrm{T}} F$, using $\log(1 + 2b^2) \leq 3 \log b$. The right-hand side is then at most $64 d_{\mathrm{T}} F$. Since $n \geq d_{\mathrm{T}}/2 > 0$, division gives $mE \leq 128F$. The dashed line to the one-sided bound follows what happens to this inequality. Solving the identity for the total remainder isolates $R_q(x)$. An estimate at integer cutoffs then gives the decay rate that extends to real cutoffs.

8 What the covolume measures[8]

A single vector has a squared norm. For several vectors the corresponding quantity is the determinant of their Gram matrix, whose entries are their pairwise inner products. For an orthogonal family this determinant is the product of the squared norms. For any family $v_1, \ldots, v_n$ it equals the squared norm of $v_1 \wedge \cdots \wedge v_n$ in the induced Hermitian norm. This squared volume is measured in the span of the family. Hadamard's inequality bounds the determinant by the same product of squared norms. In Hadamard's proof, adjoining a row to the family multiplies this upper bound by the squared length of the new row. [10]

The vectors here belong to the space of target forms. The proof measures them after a fixed linear map Θ_1. This map composes the successive theta lifts, which carry forms to forms on the next Hermitian space in the construction. The subscript 1 means that the character twists in the chain are trivial. Fix a field automorphism σ of $\mathbb{C}$ and apply it to the *coordinates* of each lifted vector. Denote the resulting vectors by $v_1, \ldots, v_n$. Pair them using the positive definite Hermitian form specified by

[8] Lean source. Base revision · New/Reduction/Lattices/TwistedLattice.lean, line 271. Base revision · New/Reduction/Lattices/TwistedLattice.lean, line 169.

the rational Gram matrix of the lattice. We say the family satisfies the *covolume bound* with parameter c if its squared volume is at most $\exp(c)$ for every σ. Equivalently,

$$\left|\det\big(\langle v_i, v_j\rangle\big)_{1\le i,j\le n}\right| \le \exp(c).$$

The order matters. A general field automorphism of $\mathbb{C}$ need not commute with complex conjugation. The definition applies σ to each vector before forming the Hermitian products. Theorem C uses this information for two automorphisms at a time to bound the absolute value of a conjugate of the original determinant. If every squared norm is at most $\exp(h)$ for every σ, Hadamard's inequality gives the uniform bound $c = nh$. Theorem C takes precisely this uniform determinant bound as a hypothesis. For an independent family of nearly full rank, it gives a lower bound on $c + 2\delta$, where δ accounts for clearing denominators. [1]

9 Theorem A(a). Short integral forms[9]

Let V be the complex space of ramified source forms and W the complex space of target forms. The source consists of algebraic forms of fixed level and weight on the initial definite Hermitian space. The ramification condition requires the translates to sum to zero over the cyclic group generated by local scaling at each designated place above the auxiliary level prime. On the target side, the general linear groups over the residue fields at the chosen places above two auxiliary level primes act on forms of principal level. The construction fixes a projection in the group algebra of their product and takes its image as the target space. A source form is *integral* if it belongs to the specified local lattice at every prime and every place above that prime. For an element g of this finite product of general linear groups, write P_g for the corresponding normalized Poincaré map. The unnormalized map forms a Poincaré series from the Bessel function associated with the source form. The operator P_g translates this series by g, projects to W, and applies the scalar normalization used in the norm and index estimates. Poincaré-series realization. One scalar comes from a seesaw identity, which expresses a pairing of theta lifts by transferring it between groups. Kudla gives the basic identity for these pairings. Wang studies a related transfer for automorphic periods, which are integrals of forms against prescribed test data. [11] [12] Write Θ_1 for the composite of the m successive theta lifts, with every character twist trivial. In the squared norm below, the subscript σ means that the field automorphism σ of $\mathbb{C}$ acts on the lifted vector's coordinates before its Hermitian norm is taken. [1]

Size estimate. The integral forms φ satisfying the following bound for every g and σ span all of V. The exponent separates three contributions. The geometric main term H measures derivative size after allowance for divisibility. The signed error E sums the deviations of the local prime sums from their main term. The explicit allowance is $F = fa(\log b)^4$, where f counts the infinite places, a is the derivative order, and b is the weight parameter. The main inequality gives the expressions for these quantities.

$$\|\Theta_1 P_g \varphi\|_\sigma^2 \le \exp(H - mE + F).$$

To see why divisibility affects this upper bound, consider one finite prime p and one infinite place. In the derivative estimate the powers p^e, $e \ge 1$, contribute according to how p factors in $\mathbb{Q}(\sqrt{-d})$, the quadratic field determined by the fixed prime d. Here $d \equiv 7 \bmod 8$. By quadratic reciprocity, for every odd prime $p \ne d$ the Legendre symbol (p/d) is 1 or -1 according as p splits or remains inert in this field. At $p = 2$ the supplementary law gives $(2/d) = 1$, and 2 splits because the field discriminant $-d$ is 1 modulo 8. A split prime contributes nothing. At an inert prime only odd powers contribute, with coefficient one. At the ramified prime $p = d$ every power contributes with coefficient one half. These three cases give exactly $\frac{1}{2}(1 - (p^e/d))$, where (p^e/d) is the Legendre symbol. Multiplying by

[9] Lean source. Base revision · New/Reduction/ShortAndEnough.lean, line 202.

$2\log p$ converts a local divisibility exponent to a logarithmic squared-norm contribution. Since the von Mangoldt function satisfies $\Lambda(p^e) = \log p$ and vanishes away from prime powers, summing these contributions gives the local prime sum in the main inequality. Local divisibility coefficients.

The divisibility exponents add along the m successive lifts. The normalized Poincaré map includes a rational factor that divides out the prime powers predicted by these exponents, allowing for the local losses and rounding to integral exponents. This division subtracts their contribution from the logarithmic squared norm. For derivatives of order a, summing the local prime sums minus their prescribed main terms over all places in class 1 gives E. In the sum of the divisibility exponents, each lift contributes this order-a sum once. Subtracting its centered value over the chain gives $-mE$. The other sum, at orders $\kappa(\tau)$ over all infinite places, occurs with multiplicity equal to the source rank at each lift. Combining that sum with the archimedean contribution gives H, while projection, normalization and rounding introduce additional errors. The proof of the short-form estimate bounds their total contribution by $x^{99} + 4fa(\log b)^3 + 2b$. The scale inequalities put this sum below F. Normalization of the squared norm.

It remains to have enough input forms on which to apply the bound. The integral source forms with bounded conjugate lengths span at the initial level. Applying the product of the differences between the identity and the local scaling translations gives the ramified subspace and preserves integrality, with a controlled increase in length. Their images thus give a spanning family to which the lift estimate applies. The main inequality selects independent vectors from the translated images of this family and applies Hadamard's inequality to their Gram determinant. Spanning by short ramified forms.

10 Theorem A(b). The cost of clearing denominators[10]

Let V be the complex space of source forms satisfying the local vanishing conditions of Theorem A(a). Fix a finite set T of group elements indexing the normalized Poincaré maps P_g of Theorem A(a). Each map translates the Poincaré image by g before projecting to the target space. At a finite place a target form can acquire denominators even when its source was integral. A bound on the logarithmic index controls the valuation of the determinant of a local matrix that clears denominators simultaneously for all admissible twists.

Index estimate. A finite family of target forms has logarithmic index at most δ if its denominators can be cleared at the following cost. At every place above a prime ℓ, an integral matrix must carry the family into the target lattice after each of the character twists allowed at ℓ. Here a twist specifies the characters in the chain of theta lifts. A twist is allowed at ℓ when none of the finite places in its modulus lies above ℓ. The same matrix must work for every such twist. Its determinant has valuation at most e_ℓ, where the nonnegative losses satisfy $\sum_{\ell\in S} e_\ell \log \ell \le \delta$ for every finite set S of primes. The local matrix may depend on the place. To state the estimate, let q be the arithmetic modulus, Φ_a the infinite places labelled by the class 1 modulo q through the action on roots of unity described in the main inequality, a the derivative order, and x_1 the larger cutoff in the prime sums. If every member of the family comes from an integral source form under a translate indexed by T, the following value of δ suffices.

$$\delta = 32\,|T|\dim(V)\,|\Phi_a|\,a\log x_1.$$

The refinement adds the local determinant losses one prime at a time. To recover the bound used in the main comparison, let f count all infinite places, and let b be the weight parameter. Since $|\Phi_a| \le f$, $\log x_1 \le \log b$ and $\log b \ge 4$, the coefficient $32|\Phi_a|a\log x_1$ is at most $2fa(\log b)^3$. [6]

The main inequality applies this estimate after choosing an independent family from the translated integral forms. The bound on the number of translates controls δ for that family, so the proof can

[10] Lean source. Base revision · New/Reduction/ShortAndEnough.lean, line 324. PR #9439 · New/Reduction/ShortAndEnough.lean, line 324.

account for the term 2δ when it compares the upper and lower volume estimates.

11 Theorem B. Spanning with few translates[11]

To use the lower bound for volume, we need independent forms spanning almost the whole target. Let V and W be the complex spaces of ramified source forms and projected target forms defined in Theorem A(a). Write $d_{\mathrm{T}} = \dim(W) > 0$. Each group element g gives a linear map P_g from V to W by translating and projecting the Poincaré image. The weight parameter b controls both the permitted number of translates and the fraction of dimensions they may miss. Theorem B gives a finite set T of group elements satisfying both estimates. [1]

$$|T| \dim(V) \le 5 d_{\mathrm{T}} \log(1 + 2b^2),$$
$$\dim\left(\sum_{g \in T} P_g(V) \right) \ge \left(1 - \frac{4}{b^2}\right) d_{\mathrm{T}}.$$

The proof first chooses the translates for the full source space. Their images miss at most $2d_{\mathrm{T}}/b^2$ dimensions. Restricting each translate to the ramified subspace loses at most the codimension of that subspace per translate. The codimension is at most b^{-8} times the dimension of the full source. For the chosen T, that dimension times $|T|$ is at most $5d_{\mathrm{T}} \log(1 + 2b^2)$. Since $5b^{-8} \log(1 + 2b^2) \le 2/b^2$ at the prescribed scale, the total extra loss is at most $2d_{\mathrm{T}}/b^2$. Adding the two losses gives the stated rank bound. Proof of the rank estimate.

The sum in the second line is the linear span of the image subspaces. Its dimension can be smaller than the sum of their dimensions because images may intersect. The first bound counts $\dim V$ for each translate of the ramified subspace, even when its image overlaps earlier images. It follows from the corresponding bound for the full source because restriction can only decrease the dimension. It bounds precisely the factor that appears in the logarithmic index estimate of Theorem A(b).

Each image is spanned by images of short integral forms, since these forms span the source. Select a basis from their union. Its members retain the individual norm estimates, and its size n is the dimension of the sum in the second line. Its span therefore has codimension at most $4d_{\mathrm{T}}/b^2$. The main inequality uses both estimates for this one choice of translates. The rank estimate makes the family large enough for the lower volume bound, while the bound on the number of translates limits the cost of clearing its denominators.

12 Theorem C. A lower volume bound for a family of nearly full rank[12]

Let $y_1, \dots, y_n$ be linearly independent elements of the space of target forms. Let d_{T} be the complex dimension of the target space and b the weight parameter. Here *nearly full rank* means that the family spans at least $(1 - 4/b^2)d_{\mathrm{T}}$ dimensions. Suppose also that the family admits logarithmic index bound δ in the sense of Theorem A(b) and covolume bound c. Thus δ bounds the logarithmic cost of clearing denominators, and c bounds the logarithm of the squared volume of the lifted family after any field automorphism of $\mathbb{C}$ is applied to its coordinates. Write H for the common geometric main term in the two height estimates, f for the number of infinite places, and a for the derivative order. With these quantities as in the main inequality, Theorem C gives

[11] Lean source. Base revision · New/Reduction/ShortAndEnough.lean, line 399. PR #9439 · New/Reduction/ShortAndEnough.lean, line 400.

[12] Lean source. Base revision · New/Determinant/Twist/LongForms.lean, line 274.

$$c + 2\delta \geq d_{\mathrm{T}} \left(H - 3fa(\log b)^4\right).$$

Complete the family to a basis by adjoining $d_{\mathrm{T}} - n$ auxiliary forms. For a choice ξ of characters in the chain of theta lifts, let Θ_ξ be the composite lift and let $G(\xi)$ be the Gram determinant of the images of this basis. The trivial choice is written 1. The factor by which the lift multiplies squared volume is $Z(\xi) = \det({\Theta_\xi}^*\Theta_\xi)$, where the star denotes the Hermitian adjoint. Multiplying $Z(\xi)$ by the Gram determinant of the original basis gives $G(\xi)$. The untwisted lift is injective, so its images of the basis are independent and $G(1) \neq 0$. Dividing the twisted identity by the untwisted one cancels the common Gram determinant of the original basis.

$$\frac{G(\xi)}{G(1)} = \frac{Z(\xi)}{Z(1)}.$$

At a prime ℓ, fix an abstract field isomorphism $\iota : \mathbb{C} \to \overline{\mathbb{Q}}_\ell$. The ℓ-adic valuation of $\iota(z)$ measures the denominator of a coefficient z. The twist selection theorem supplies one twist at each such identification. Its modulus is coprime to ℓ. The theorem makes these choices simultaneously so that, at each of the m lifts, the family of untwisted-to-twisted determinant ratios has logarithmic height at most $d_{\mathrm{T}}(F-H)/m$, where $F = fa(\log b)^4$. This height bound can be negative because its logarithmic contribution at infinity can be negative. The height calculation below explains how it combines with the contributions from the finite places. The index estimate clears denominators for the original forms for both ι and the map $z \mapsto \iota(\bar{z})$, since the same twist is allowed at both. The two determinant losses account for 2δ. For the chosen twist, one further scalar clears the auxiliary forms at both identifications.

To use $G(1)$ as a common multiplier, one also needs an upper bound for $|\sigma(G(1))|$ for every automorphism σ of $\mathbb{C}$. Let w be the exterior product of the untwisted lifts of the completed basis, so $G(1) = \langle w, w\rangle$. Define $\sigma'(z) = \overline{\sigma(\bar{z})}$. The matrix of the Hermitian form has rational entries, so applying σ to the scalar $\langle w, w\rangle$ gives $\langle \sigma' w, \sigma w\rangle$, with both automorphisms acting on coordinates. Cauchy–Schwarz now gives

$$|\sigma(G(1))| \leq \|\sigma' w\| \, \|\sigma w\|.$$

Each squared norm satisfies the same covolume bound for the completed basis. Their geometric mean therefore gives that bound for the absolute value of the conjugated determinant. Bound for conjugates of the Gram determinant.

At each finite place multiplication by $G(1)$ turns $Z(\xi)/Z(1)$ into $G(\xi)$ by the displayed identity. This is precisely the local determinant controlled by clearing denominators. Gram determinant comparison. [15]

To pass from the reciprocal ratio for one lift to the ratio for the entire chain, factor the chain determinant into the single-lift determinants and a scalar correction. Each single-lift estimate contributes $d_{\mathrm{T}}(F-H)/m$ to the logarithmic bound. Each uses the same sum of derivative sizes less divisibility over the infinite places that occurs in Theorem A(a). Summing over the m lifts therefore gives precisely $d_{\mathrm{T}}(F-H)$. The scalar correction costs at most $d_{\mathrm{T}}b$. Single-lift and chain estimates.

The twist may depend on the prime and on the identification ι. A height bound for such a family uses one nonzero scalar for all identifications. It bounds the logarithmic absolute value of the scalar under every automorphism of the complex numbers and, for each prime, the logarithmic absolute values of the scaled local entries uniformly over the identifications at that prime. The archimedean bound plus the sum of the nonnegative local allowances over any finite set of primes must fit within the stated height bound. Multiplying the two families multiplies their scalars and adds their bounds. At each identification the forward and reciprocal ratios use the same twist and multiply to 1, since the chosen lifts are injective. For the constant family, the local bound at 2 bounds the 2-adic absolute value of the common scalar under *every* identification. The algebraicity criterion used here therefore

makes the scalar algebraic over $\mathbb{Q}$. Its field norm is a nonzero rational number. Applying the product formula to that norm shows that the combined height bound is nonnegative. This is how the locally chosen twists enter one global inequality. Height of the constant family. [13]

Let x_1 be the larger cutoff in the prime sum. Each auxiliary form costs at most $2x_1^2b^2$ in logarithmic squared norm and b for its clearing scalar. The forward ratio thus has logarithmic bound $c + 2\delta + (d_{\mathrm{T}} - n)(2x_1^2b^2 + b)$. Adding the reciprocal bound $d_{\mathrm{T}}(b + F - H)$ gives a nonnegative sum by the product formula. Rearranging gives

$$d_{\mathrm{T}}(H - F - b) \le c + 2\delta + (d_{\mathrm{T}} - n)(2x_1^2b^2 + b).$$

Near-full rank gives $d_{\mathrm{T}} - n \le 4d_{\mathrm{T}}/b^2$. The scale inequalities give $4(2x_1^2 + 1/b) \le b$ and $b \le F$. The entire completion thus costs at most $d_{\mathrm{T}}b$, and the two losses of $d_{\mathrm{T}}b$ fit inside $2d_{\mathrm{T}}F$. This yields the stated bound. The main inequality applies it to the same family whose individual norms give an upper bound for c, so the two estimates can be compared directly. Completion estimate and Theorem C.

Edition and scope

The base text is the community's readable edition of Levent Alpöge's formalization, at the November 27 revision cited below. The diagram restricts the compiled dependency relation, including references in types and proof bodies, to its twelve named declarations. The map shows the transitive reduction on these twelve declarations. For all 132 ordered pairs of distinct nodes, following lines in the map gives exactly the same reachability as following references in the compiled edition. Thirty-four pairs are reachable. The horizontal positions are chosen for legibility. The exposition incorporates three pending revisions affecting these statements and their final specialization. In particular, the sharper index estimate implies the coarser estimate needed in the fourth-power argument. We reconciled the two estimates and the modulus-one specialization in one local build on Lean 4.37.0-rc1. That build recompiled the Riemann hypothesis in both of its displayed formulations, the zero-free half-plane theorem for Dirichlet characters, and the critical-line theorems for all characters and for primitive characters. Each depends only on propositional extensionality, classical choice and the quotient soundness axiom. The reachability relation among the twelve displayed declarations is the same on the upstream main branch and in this edition. Changes inside the omitted paths can still matter mathematically. The source pane gives the local declaration and labels upstream links by revision. This edition incorporates the three proposals affecting the displayed theorem statements or their specialization; the other proposals are outside the joint build. [1] [5] [6] [7]

Proof compression

Several community revisions shorten proofs by sharing an intermediate calculation. Examples include changing coordinates in a Gram matrix, adjoining two families of forms, and showing that two local correction terms vanish at the same primes. [3] [4] [14] I developed Flow Proof Optimizer to help find such opportunities. Given a compiled Lean project and a target theorem, it traces the theorem's dependencies and finds repeated derivations. For supported patterns it can extract a shared lemma and check it at the original proof sites. I describe the method in the Atomic Inference article How to Simplify a Lean Proof Without AI.

These accounts contributed to shortening or sharing Lean proofs in the community repository. The contributors appear in GitHub's contributor order for the period beginning August 22, 2026.

sqrt-of-2, claude, weiwensangsang, Deicyde, roy7, CollinYuanjieRen, dilarasoylu, STARDUSTLC666, Liangebra, ashleyholman, kim-em, dga471, alkor2000, rohanarun, CubeJerry, DocLoJag, riccardobrasca, TitiSkywalker, Dharmavati213, yidakra, josephch405, chopin-op60, jasonyandell, Eric-He-98, pranavgundu, cannonballgamesstudio, maaz7409, sangxsh, nefelibata5136-code, codex, koefoeden, gmberton, hfef7ui2, deancureton, CLaSLoVe, homeowmorphism, danielkwalsh, josephlr, HaraldNordgren, bohemian-miser, TmLev,

WinglokWong, dishank-hai, righi, uos1231234, andersonwu2000, AahilShaikh, thekeunpie-hash, rmsnorm, imvladikon, LizzzMax, zyz758, nickita-khylkouski, eumemic, aviadr1, Grativy6, RudiksChess, lutrarutra, amilcodes

Last updated: November 27, 2026 at 2:37 PM EST

If you're a mathematician interested in improving this reading guide, please contact me via this email. The base text and proposed changes are available in the Lean community's Riemann hypothesis repository.

References

1. L. Alpöge with Claude; Lean community readable edition. *Riemann hypothesis and GRH for Dirichlet L-functions.* Revision 8c828eda, November 27, 2026. Reading guide and source.

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3. Y. Ren. Two proposed changes of basis in the determinant argument. Volume preservation under Gram coordinate changes (PR 9630, 9878f40a) and equal inverse rows (PR 9662, 046fcd0f).

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5. imvladikon. "Keep the fourth power in the main inequality." PR 5659, revision 681d6949. Main inequality and its analytic consequence.

6. Grativy6. "Sharpen the logarithmic index bound in Theorem A(b)." PR 9439, revision 94057773.

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14. uos1231234. "Share the vanishing of gains and losses outside their support." PR 6711, revision 28777078.

15. andersonwu2000. “Bound the forward volume ratio of a twisted lattice.” PR 7373, revision 8f02ba5b.

Cite this blog

If you refer to this article in your work, please use the following BibTeX entry.

```
@misc{Ren2026ReadableRH,
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title = {A Guide to Reading the {Riemann} {Hypothesis} Proof in {Lean}},
year = {2026},
url = {https://collinyuanjieren.com/blog/readable-RH/}
}
```

Notation mapping

The formulas use mathematical names for the objects in the local reading edition. In the Lean expressions, `P` fixes the arithmetic parameters and `D : New.Data P` fixes the geometric data. A projection such as `D.f` denotes `New.Data.f D`. The article suppresses these fixed arguments. For example, `New.primeRemainder d q y` becomes $R_q(y)$: the subscript names the modulus, the argument is the cutoff, and the prime is held fixed.

Symbols and their scope

q_1 `D.q₁`
Order of the root of unity adjoined to the quadratic base field. It is a multiple of q. Main inequality for the total remainder.

κ `D.weight`
Derivative order assigned to an infinite place. Main inequality for the total remainder.

s_{fin} `D.primeSum`
Local weighted prime sum at derivative order ν and infinite place τ. Main inequality for the total remainder.

Θ_ξ `New.Data.Θ D ξ`
Composite theta lift for the chain of characters ξ. Theorem C. A lower volume bound for a family of nearly full rank.

Z `New.Data.chainDeterminant D`
Squared volume factor for the composite theta lift as a function of its twist. Theorem C. A lower volume bound for a family of nearly full rank.

G `G in New.TwistedLattice.heightLe_volume_div`
Gram determinant for the lifted completed basis as a function of its twist. Theorem C. A lower volume bound for a family of nearly full rank.

$\widehat{R}_q$ `New.isPolar_primeRemainder`
Article notation for the meromorphic continuation of the integral of the weighted prime remainder against y to the power −s. The prime d is fixed. Zero-free criterion from a one-sided remainder estimate.

R_q `New.primeRemainder d q`
Weighted prime remainder as a function of a real cutoff. The subscript records the modulus. The prime d is fixed. Weighted prime remainder.

V `New.Data.ramifiedSourceForms D`
Complex space of source forms satisfying the local vanishing conditions. Theorem A(a). Spanning by short integral forms.

W `New.Data.targetForms D`
Complex space obtained by projecting the target forms at principal level. Theorem A(a). Spanning by short integral forms.

d_{T} `New.Data.targetDim D`
Complex dimension of W, the target space of forms. The subscript T stands for target. Theorem B. A nearly full span with few translates.

P_g `New.Data.normalizedPoincare D g`
Normalized Poincaré map from V to W for the translate indexed by g. Theorem A(a). Spanning by short integral forms.

E `New.Data.totalRemainder D`
Signed sum of the errors in the local prime sums over the infinite places in class 1. Main inequality for the total remainder.

H `New.Data.mainAmount D`
Geometric main term shared by the upper and lower height estimates. Main inequality for the total remainder.

$fa(\log b)^3$ `New.Data.errorUnit D`
Cubic logarithmic allowance used to bound the cost of clearing denominators. Theorem A(b). Logarithmic index bound.

M `D.M`
Principal level in the archimedean normalization of the Taylor coefficients. Main inequality for the total remainder.

Ω `P.bigOmega D.towerExp`
Positive real period in the archimedean normalization. Main inequality for the total remainder.

k `D.k`
Rank of the Hermitian source space in the formula for H. Main inequality for the total remainder.

α δ `in zero_density_quadTwist`
Exponent $(\beta - 1/2)/9$ in the exceptional-prime count. Zero-free criterion from a one-sided remainder estimate.

T `T in zero_density_quadTwist`
Square root of qX when counting exceptional primes up to X. Zero-free criterion from a one-sided remainder estimate.

R_1 `New.primeRemainder d 1`
Modulus-one specialization, with the same fixed prime d as in the remainder at modulus q. Eventual one-sided bound for the prime remainder.

c_q `(New.logDerivMean d q 1).re`
Real normalizing constant subtracted in the prime sum. It also depends on d. Weighted prime remainder.

q `P.q`$_0$
Arithmetic modulus in the geometric construction. In the half-plane argument the auxiliary modulus is 18480 times the original character modulus. Zero-free half-plane for Dirichlet L-functions.

d `P.d`
Prime specifying the quadratic character used in the weights of the prime sum. Weighted prime remainder.

x `D.x`
First cutoff in the prime sum. Eventual one-sided bound for the prime remainder.

x_1 `D.x`$_1$
Larger cutoff at which the remaining arithmetic errors are estimated. Eventual one-sided bound for the prime remainder.

b `D.b`
Weight assigned to the selected infinite places in class -1. Main inequality for the total remainder.

a `D.a`
Derivative order, the product of the first cutoff and the weight. Main inequality for the total remainder.

m `D.m`
Number of successive theta lifts in each chain. Theorem A(a). Spanning by short integral forms.

f `D.f`
Total number of infinite places of the auxiliary CM field. Main inequality for the total remainder.

Φ_a `D.aPlaces`
Set of infinite places in class 1 modulo q. Main inequality for the total remainder.

Φ_b `D.bPlaces`
Selected subset of the infinite places in class -1 modulo q. Main inequality for the total remainder.

Θ_1 `New.Data.Θ D 1`
Composite theta lift with every character twist trivial. Uniform bound on the logarithm of the squared covolume.

F `sizeError`
Local abbreviation in the proof of the main inequality for D.f * D.a * D.logB ^ 4. The fourth power occurs in the height estimates. Main inequality for the total remainder.

T `t`
Finite set of group elements indexing the selected translates. In the index estimate this set is an argument. In Theorem B it is chosen by the theorem. Theorem B. A nearly full span with few translates.

δ `D.targetLattice.IndexLe y δ`
The final argument bounds the logarithmic cost of clearing denominators for the family y. Theorem A(b). Logarithmic index bound.

c `D.targetLattice.CovolumeLe y c`
The final argument bounds the logarithm of the squared covolume of y, uniform over complex field automorphisms. Uniform bound on the logarithm of the squared covolume.

Declarations in the proof map

Riemann hypothesis
`riemannHypothesis` `Main.lean` · L9

Statement locating the nontrivial zeta zeros
`RiemannZetaZerosOnLine` `Statements.lean` · L7

Zero-free half-plane for Dirichlet L-functions
`New.grh` `New/Reduction/OneSidedBound.lean` · L249

Zero-free criterion from a one-sided remainder estimate
`New.LFunction_ne_zero_of_primeRemainder_le` `New/Reduction/Zeros/Criterion.lean` · L41

Weighted prime remainder
`New.primeRemainder` `New/Reduction/Zeros/PrimeError.lean` · L74

Eventual one-sided bound for the prime remainder
`New.eventually_primeRemainder_le_mul_rpow` `New/Reduction/OneSidedBound.lean` · L213

Main inequality for the total remainder
`New.star_inequality` `New/Reduction/MainInequality.lean` · L93

Uniform bound on the logarithm of the squared covolume
`New.TwistedLattice.CovolumeLe` `New/Reduction/Lattices/TwistedLattice.lean` · L271

Theorem A(a). Spanning by short integral forms
`New.Data.span_short_integralRamified_eq_top` `New/Reduction/ShortAndEnough.lean` · L202

Theorem A(b). Logarithmic index bound
`New.Data.indexLe_normalizedPoincare` `New/Reduction/ShortAndEnough.lean` · L324

Theorem B. A nearly full span with few translates
`New.Data.enough_forms` `New/Reduction/ShortAndEnough.lean` · L400

Theorem C. A lower volume bound for a family of nearly full rank
`New.Data.integral_forms_long` `New/Determinant/Twist/LongForms.lean` · L274